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Orientifold planes, affine algebras and magnetic monopoles

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Abstract: We analyze string theory backgrounds that include different kinds of orientifold planes and map out a natural correspondence to (twisted) affine Kac-Moody algebras. The low-energy description of specific BPS states in these backgrounds leads to a construction of explicit twisted magnetic monopole solutions on $\mathbb{R}^3 \times S^1$. These backgrounds yield new low-energy field theories with twisted boundary conditions and the link with affine algebras yields a natural guess for the superpotentials of the corresponding pure $\mathcal{N} = 1$, and $\mathcal{N} = 1^*$ gauge theories.

Keywords: Solitons Monopoles and Instantons, D-branes, Gauge Symmetry.

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Contents

1. Introduction

String theory backgrounds including orientifold planes have proven to be a fruitful ground for study. Since their construction as perturbative string backgrounds [1, 2, 3], (see [4] for a recent review on the perturbative aspects of these objects) much has been learned on their non-perturbative aspects. See [5] for a recent review, and further developments can be found in [6, 7, 8]. New types of orientifold planes are discovered by turning on various discrete fluxes, changing the charges and tensions of these objects. By employing a general principle of branes ending on branes, it is possible to introduce "brane realizations of Discrete Torsion" [5]. This allows for a simple intuitive understanding of the various types of orientifold planes present in the string theory backgrounds. Especially interesting is the classification of disconnected components on the moduli space of compactifications for a small number of compact dimensions. For backgrounds with 16 supercharges this question received some answers [7]–[10].

Such backgrounds are partially classified by the rank of the gauge group,[1] alternatively by the number of massless vector multiplets in the low energy theory. Let us review some of these results. In 10 dimensions there are 2 theories, each with rank 16 and with a single scalar in a gravity multiplet (dilaton). One is the $Spin(32)$ theory and the other the $E_8 * E_8$. No continuous parameter connects these theories in 10 dimensions. In 9 dimensions there are 3 disconnected components, with ranks 17, 9, 1. These, for example, correspond to M theory compactifications on a cylinder, Möbius strip and Klein bottle, respectively. In 8 dimensions we have 3 components with ranks 18, 10, 2, etc. The analysis can be extended down to very low dimensions as recently studied in [12]. The present paper, however, will follow a different route.

As for some of the low rank cases above, we will be interested in studying backgrounds including different types of orientifold planes. The mere fact that D-branes placed in this background have different enhanced gauge symmetries, depending on their location relative to the different orientifold planes, leads to complications in formulating the low-energy theory on these D-branes. For example, in a background which contains an O^+ plane and an O^- plane, the enhanced gauge group is Sp near the O^+ plane and is SO near the O^- plane. The gauge theory on the D brane can be written, locally, at the vicinity of each of the orientifold planes, but not in a global fashion which will take both gauge enhancements into account. A clue to resolving this issue comes from analyzing certain BPS-states that exist in these backgrounds – this lays bare a natural connection to affine algebras (see e.g. [13, 14] and references therein for earlier analysis of infinite algebras in the context of string theory). In this paper we show that this analysis leads to a stringy realization of (twisted) affine algebras, and periodic instanton and monopole solutions in low-energy theories with twisted boundary conditions.

The outline of the paper runs as follows. In section 2, we review the guise in which monopoles appear in D-brane settings including orientifold planes. In section 3 we combine the building blocks from section 2 to make a connection between orientifold planes and affine algebras. Here we present a correspondence between a set of affine and twisted affine algebras and their corresponding representations in terms of branes and orientifolds. While the construction is very simple, it gives a very powerful realization of infinite dimensional algebras on branes in string theory. In particular, we point out that the introduction of a compact direction transverse to a set of D branes turns the finite dimensional gauge symmetry on the branes to an infinite dimensional one. In section 4 we show how the enhanced gauge symmetries at the orientifold planes are encoded in the algebraic framework. We study various limits on the moduli space and find agreement with the string theory expectations. Section 5 is devoted to finding explicit descriptions of BPS states in the low-energy field theory and section 6 contains conclusions. In appendix A we summarize our conventions.

[1]Rank reduction was first discussed in [11].

2. Review of D-branes and monopoles

In this section we review the relation between BPS monopoles and D-branes in a simplified setting, and introduce the building blocks that are needed in the next section for a more detailed analysis of less well-known string theory backgrounds.

A string vacuum consisting of N parallel, separated and co-linear $D3$-branes (figure 1), preserving 16 supercharges, contains non-perturbative BPS states that correspond to $D1$-branes stretching between $D3$-branes. In the low-energy field theory on the $D3$-branes these correspond to magnetic monopole solutions in a spontaneously broken gauge theory [15, 16, 17], where the scalar vacuum expectation value at infinity sets the separation between the $D3$-branes.

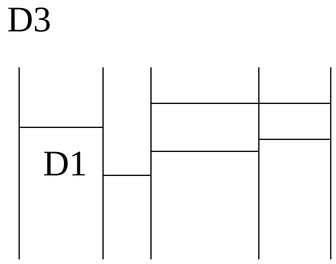

Figure 1: $D3$-branes separated on a line with $D1$-branes stretching between them.

The fundamental monopole solutions [18] (in the background of which the Dirac operator has only four zero modes) correspond in figure 1 to $D1$-branes stretching between two neighboring $D3$-branes. On the other hand, it is intuitively clear that a $D1$-brane stretching between two $D3$-branes that are not neighbors will have more zero modes since it can split into one or more $D1$-branes (see the right hand side of figure 1). For such a case the corresponding monopole solution is not fundamental. The fundamental monopole solutions can be labeled by the simple co-roots of the gauge group [18] associated to the $D3$-branes. For our $\mathrm{SU}(N)$ example[2], there are $N-1$ simple co-roots, in one to one correspondence with $D1$-branes stretching between neighboring $D3$-branes. S-duality leads to similar reasoning for the electrically charged macroscopic $F1$-strings. These are associated with the roots of the gauge group.

When we include orientifold planes (the orientifold planes which will be considered are the O^-, O^+, $\widetilde{O}^-$ and $\widetilde{O}^+$, as reviewed in [5]) in the string background, the analysis changes slightly. In the presence of orientifold planes, some states are projected out of the string spectrum, and the analysis is repeated taking into account that only certain $D1$-branes survive as BPS states in the spectrum. In the S-dual picture, which admits a perturbative analysis and is more convenient for our purposes, we can understand the absence of certain states as due to the orientifold projection condition [19, 20] that yields only a subset of invariant $F1$-string states. In figure 2 we depict the covering space and draw some of the fundamental $F1$-string states. The difference between the string passing the O^+ plane and the string ending on the $\widetilde{O}^-$ plane is indicated by their relative length. Note, for example, that an $F1$-string can end on an $\widetilde{O}^-$ plane, since there is a D-brane stuck on that orientifold plane, while it can not end on an O^+ plane.

[2] We ignore the center of mass $U(1)$ for now.

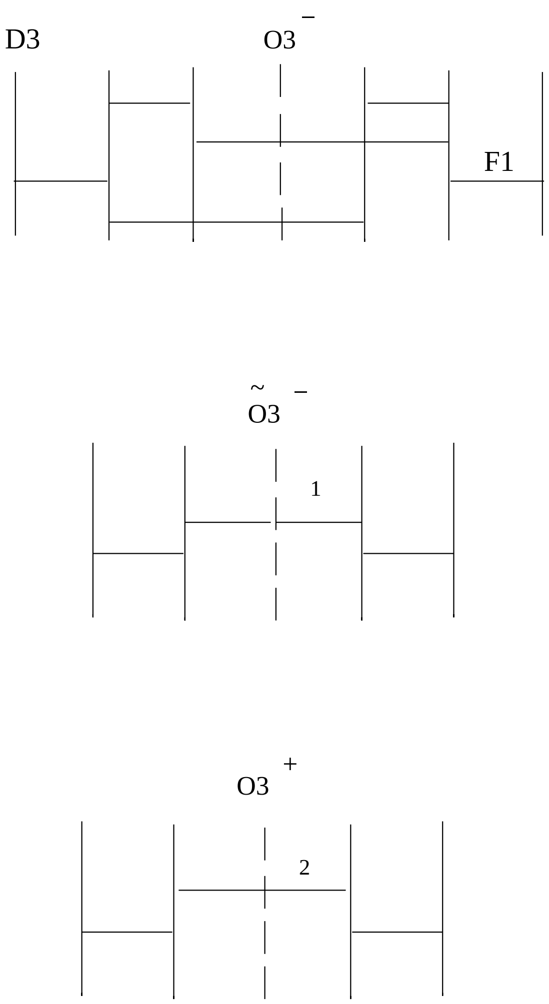

Figure 2: The three types of orientifold planes and allowed fundamental $F1$-string BPS states in the covering space. The relative length of the $F1$-strings in the cases of the $\widetilde{O3}^-$ and $O3^+$ is indicated above the $F1$-strings. Note that mirror images of the horizontal strings represent the same state (and that only the string in the middle of the lowest figure is its own mirror image). The $\widetilde{O3}^+$ plane has the same electric spectrum as the $O3^+$ plane.

The simple roots corresponding to the fundamental electrically charged solutions [18], can be represented by the distances between the D-branes on which they end. Their natural intersection form is represented by a Euclidean intersection form for the positions of these D-branes. For example, we can consider the fundamental $F1$-strings in figure 3 (where we draw the space resulting after the orientifold projection) for the $O3^-$ case. They can be associated to the positions of the $D3$-branes on which they end. We find the vectors: $e_3 - e_2, e_2 - e_1$ and $e_2 + e_1$, which have intersections proportional to the intersections of the simple roots corresponding to the nodes drawn in figure 3.

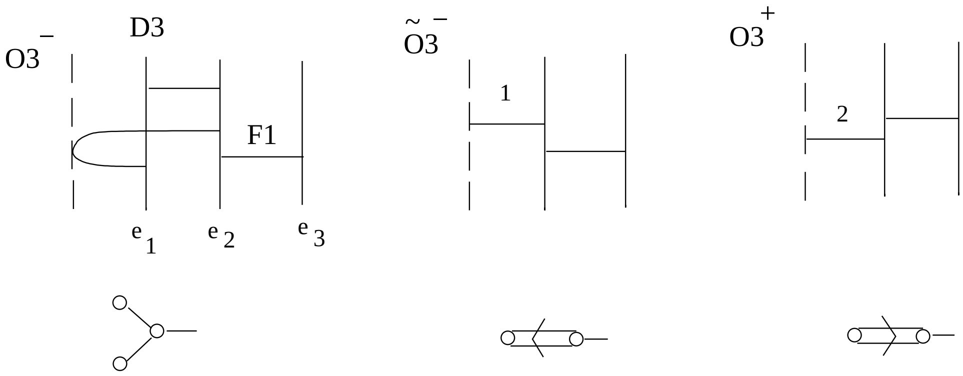

Figure 3: The building blocks for Dynkin diagrams and their corresponding orientifold planes. We indicated the positions of the $D3$-branes in the $O3^{-}$-plane example.

We can summarize the correspondence between orientifold planes combined with D-branes, and the simple root system by associating (part of) a Dynkin diagram to each orientifold plane as in figure 3. The root system is associated to the electrically charged states in the corresponding low-energy gauge theory. The electrically charged BPS states in the case of the $\widetilde{O}^{+}$ plane span the same root system as for the O^{+} plane. We return to the difference between these two cases in the next section. It resides in the magnetically charged states that correspond to a co-root system. The example we gave in this section in terms of $O3$-planes, $D3$- and $D1$-branes can be T-dualized to other $O(p+2)$-planes, $D(p+2)$ and Dp systems for $p \leq 6$, in the cases where such objects exist.

3. Orientifold planes and affine algebras

In the previous section we reviewed the building blocks that are put to good use in the present section. We combine the blocks by studying string theory backgrounds including parallel orientifold p-planes and Dp-branes. The background geometries we study include an interval with two orientifold planes at the endpoints, each can be of type $O^{-}, O^{+}, \widetilde{O}^{-}$ or $\widetilde{O}^{+}$. In between the orientifold planes we place Dp-branes, aligned with the two orientifold planes. In the covering space we have two orientifold planes on the fixed points of a circle under the orientifold action together with D-branes and their mirrors.

Using the building blocks introduced in section 2 and their magnetic associates, we can already press home one of the main points of our paper. First of all, concentrating on the electrically charged states, we can construct the Dynkin diagrams corresponding to the simple roots associated to the $F1$ branes stretching between the Dp-branes and the orientifold planes. We draw these Dynkin diagrams in fig-

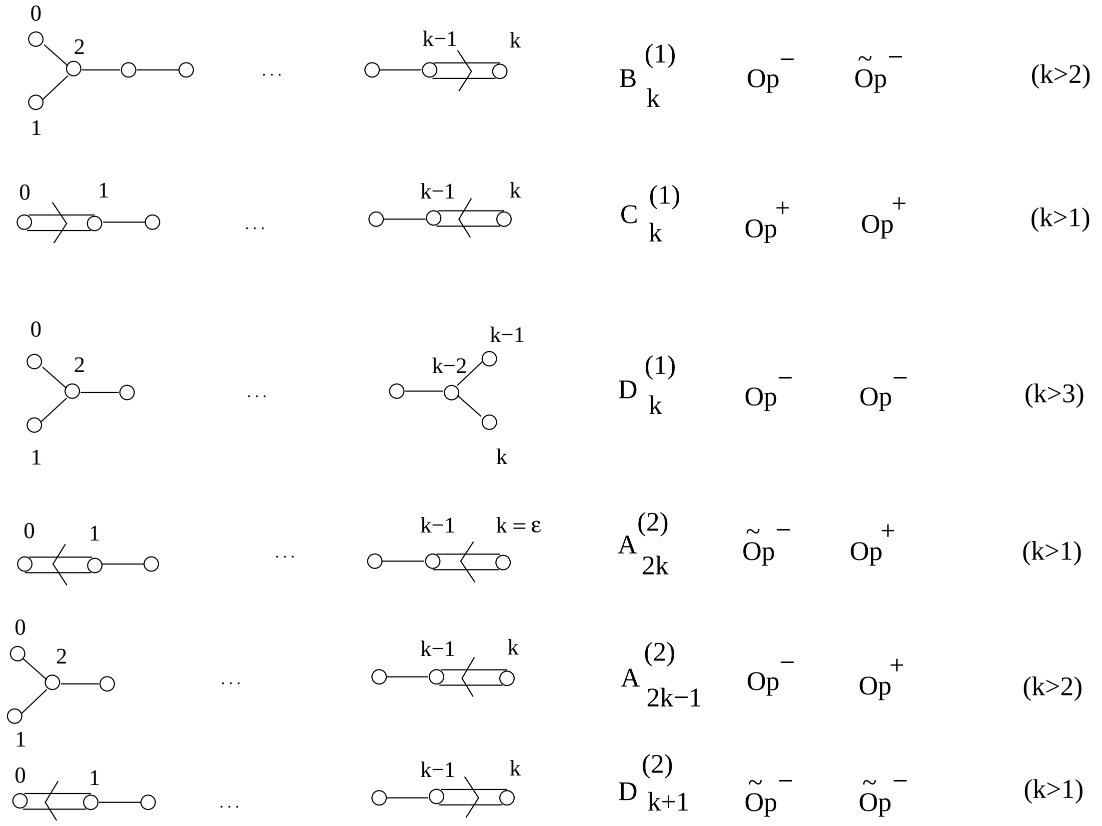

Figure 4: On the far left we denote the Dynkin diagrams of untwisted and twisted affine Kac-Moody algebras. The first 3 diagrams represent affine algebras while the last 3 represent twisted affine algebras. Their standard notation is next to the diagrams, where the superscript in the notation for the (twisted) affine algebras denotes the order of the outer automorphism that is used to construct them [21] (see figure 5). The six different combinations of the orientifold planes Op^-, $\widetilde{Op}^-$ and Op^+ used to construct the Dynkin diagrams within string theory, is shown in the next column. On the far right are limitations on the rank of the affine algebras for the Dynkin diagram to make sense. The zeroth root always corresponds to the root α_ϵ to be defined later, except for the case $A^{(2)}_{2k}$.

ure 4.[3] We recognize these Dynkin diagrams as corresponding to six infinite series of (twisted) affine algebras [21] (see also [22, 23, 24]). This allows us to put forth a correspondence between string theory backgrounds including two orientifold planes at the ends of an interval, and (twisted) affine algebras, as displayed in figure 4. We will make the nature of the correspondence more precise as we go along.

If in addition to electrically charged objects we consider magnetically charged BPS objects, we can refine our correspondence to make a distinction between the O^+ plane and the $\widetilde{O}^+$ plane. The difference between O^+ and $\widetilde{O}^+$ arises in their monopole

[3] Some low-rank examples are shown in figure 6.

planes	electric	magnetic
$--$	DD	DD
$++$	CC	BB
$\tilde{-}\tilde{-}$	BB	CC
$\tilde{+}\tilde{+}$	CC	CC
$-+$	DC	DB
$-\tilde{-}$	DB	DC
$-\tilde{+}$	DC	DC
$+\tilde{-}$	CB	BC
$+\tilde{+}$	CC	BC
$\tilde{-}\tilde{+}$	BC	CC

Table 1: On the left of the table we write the type of orientifold p-planes at the two ends of the interval, in the two following columns the algebras associated to the electrically and magnetically charged BPS states. The letters indicate a B, C or D type ending in the Dynkin diagram. For example, DB stands for the $B_k^{(1)}$ algebra.

spectrum, i.e. in the magnetically charged objects.[4] Similarly as in figure 3 (where the case of the O^+ and $\widetilde{O}^+$ coincide), we can draw a Dynkin diagram associated to the magnetically charged states. Then the (ending of the) Dynkin diagram of the O^+ plane and the $\widetilde{O}^-$ plane get interchanged, but the $\widetilde{O}^+$ plane remains associated with a C-type ending. Making use of this reasoning based on S-duality (and of an obvious notation indicating the three types of Dynkin diagram endings), we can summarize the correspondences between the ten possible combinations of orientifold planes and electric and magnetic charges in table 1.

The appearance of affine algebras in the context of compactification on a circle or an interval is no surprise. Returning to our previous basic example of $D3$-branes (without orientifold planes) with $D1$-branes stretched between them, we notice that for $D3$-branes on a circle, there appears a new fundamental magnetic monopole that stretches between the last $D3$-brane and the first, completing the circle. It is well known [26] that we can associate the extra fundamental monopole to the extra root of the affine algebra $A_{N-1}^{(1)}$, thereby closing up the Dynkin diagram of that algebra to form a necklace. Notice that we have hereby completed a realization of the infinite series of affine algebras, consisting of $A_k^{(1)}, B_k^{(1)}, C_k^{(1)}, D_k^{(1)}, A_{2k}^{(2)}, A_{2k-1}^{(2)}, D_k^{(2)}$ [21] in string theory backgrounds. Apart from these, there are a few special low-rank and exceptional cases that we will briefly return to.

4. Gauge enhancements and subalgebras

In this section we initiate a more detailed description of the link between the orientifold backgrounds and the (twisted) affine algebras. In the following, we concentrate on the electrically charged states. We will also make good use of the T-dual of these configurations along the interval separating the orientifold planes. After T duality a pair of orientifold planes combines into a single object which can be either an ordinary orientifold or a more exotic object. The results of T-duality are summarized in table 2.

[4] Restricting to $p = 3$, under S-duality the $\widetilde{O}^+$ plane is invariant. The complexified gauge

	O^-	O^+	$\widetilde{O}^-$
O^-	$O^-[D_k^{(1)}]$	$o[A_{2k-1}^{(2)}]$	$\widetilde{O}^-[B_k^{(1)}]$
O^+	$o[A_{2k-1}^{(2)}]$	$O^+[C_k^{(1)}]$	$\widetilde{o}[A_{2k}^{(2)}]$
$\widetilde{O}^-$	$\widetilde{O}^-[B_k^{(1)}]$	$\widetilde{o}[A_{2k}^{(2)}]$	O^- + a D-brane$[D_{k+1}^{(2)}]$

Table 2: "Addition table" for orientifolds. On the top (left) of the table we write the type of orientifold p-plane which is placed at the first (second) end of the interval. They combine after T-duality along the interval to an orientifold $p+1$-plane that is written in the body of the table. We also include the correspondence to affine Kac-Moody algebras in square brackets.

The notations $\widetilde{O}^-, O^+, O^-$ indicate familiar orientifold $p+1$ backgrounds, and $o, \widetilde{o}$ indicate backgrounds including modding out by a shift (more detailed in [27]–[31]). Concentrating on the algebra associated to the electric charges, we include the correspondence between the orientifold planes and the affine Kac-Moody algebras (see table 2 and figure 4). Since the occurrence of twisted affine algebras is new, we discuss especially the properties of the string backgrounds corresponding to the twisted affine algebras in more detail in the following. The combinations of orientifold planes for which the T-dual corresponds to a well-known string theory background are (O^-, O^-), (O^+, O^+) and $(O^-, \widetilde{O}^-)$. We do not treat these cases in detail. One can make a similar but more straightforward analysis, analogous to the cases discussed in detail in the next sections.

4.1 (O^+, O^-) and $(O^+, \widetilde{O}^-)$ orientifolds

In this subsection we analyze how the finite subalgebras of the affine algebras surface as enhanced gauge symmetries in the string backgrounds. We show how they arise naturally from the vector modes in the open string sector when we assume the action of an outer automorphism on the Chan-Paton matrices of the open strings.[5] We show that this assumption is consistent with our intuition for enhanced gauge symmetries near orientifold fixed planes.

First, we discuss in detail the case of the (O^+, O^-) and $(O^+, \widetilde{O}^-)$ orientifolds. We study the T-dual of these configurations, which correspond to o- and $\tilde{o}$-planes (see table 2). The orientifold action includes in this case a shift along the interval. We concentrate on the open string modes in these backgrounds since we do not expect new issues to arise in the closed string sector. The bulk closed string theories for these backgrounds are the usual Type II theories (for $p \leq 9$).

In the open string sector, we first concentrate on the massless vector states in the NS sector, associated to the $D(p+1)$-branes coinciding with the o-plane. We

coupling and theta angle undergo the change $\tau \to \tau + 1$, when the discrete flux that shifts O^+ to $\widetilde{O}^+$ is turned on.

[5]This is reminiscent of the construction of CHL strings in [32, 33].

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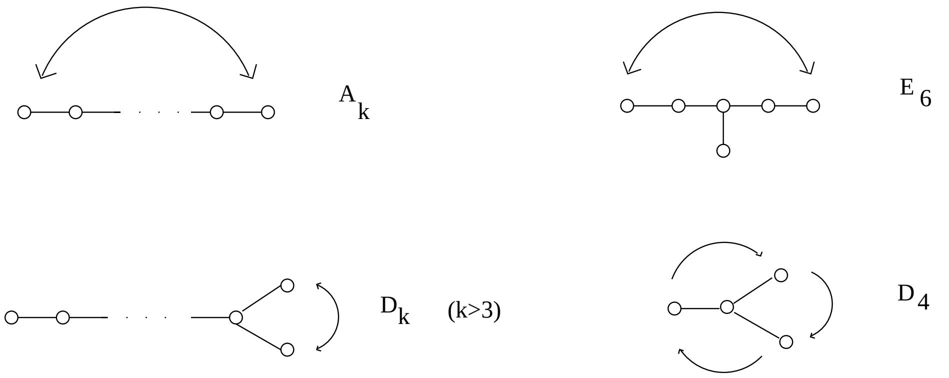

Figure 5: The action of outer automorphisms on the simple roots of Lie algebras, pictured here as an action on the Dynkin diagram, from which the action on the full algebra follows. We also displayed two special low-rank cases on the right.

mod out the vector states in the covering theory by S, the shift over πR (where R is the radius of the covering circle), combined with Ω (orientation reversal on the worldsheet), but also with the action of an outer automorphism τ on the Chan-Paton factors, as follows,

$$\psi^{\mu}_{-1/2}|x\rangle_a T^a = +\psi^{\mu}_{-1/2}|x+\pi R\rangle_a \tau(T^a)\,, \tag{4.1}$$

where ψ is the worldsheet fermion in the NS sector, the outer automorphism acts by reflecting the A_{N-1} Dynkin diagram, and where N is the number of D-branes in the covering theory (which is even for the (O^+, O^-) case and odd for the $(O^+, \widetilde{O}^-)$ case). Figure 5 shows the action of the outer automorphism on the Dynkin diagram.

The index μ is along the $D(p+1)$ worldvolume and we have $\tau^2 = 1$. This is our proposal for the action of the orientifold operation on the open string states in the presence of an o-orientifold, and we will show it to be consistent with intuitive expectations.

To describe the open string vector states after modding out, we define the following subalgebras of the parent A_{N-1} Lie algebra. We distinguish the g_0-subalgebra (with elements T_0^a) corresponding to linear combinations of generators in the Lie algebra with eigenvalue 1 under the outer automorphism, and the g_1-subalgebra (with elements T_1^a)[6] corresponding to generators with eigenvalue -1. After modding out, we find the following open string states on the worldvolume of the $D(p+1)$-branes in the presence of the o-orientifold:

$$\psi^{\mu}_{-1/2}|2k\rangle_a T_0^a \quad \text{and} \quad \psi^{\mu}_{-1/2}|2k+1\rangle_a T_1^a\,, \tag{4.2}$$

[6]The notation for the subalgebras is standard [21, 22].

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where k is an integer specifying the momentum of the open string state[7]. The parity of the momentum is paired with the eigenvalue under the outer automorphism to insure invariance under the combined orientifold operation (4.1).

Note that we have not yet turned on any Wilson lines in the compact direction on the worldvolume of the $D(p+1)$-branes. The massless vector states on the worldvolume of the branes correspond to the modes with zero momentum. They generate the g_0-subalgebra of the twisted affine algebra $A_{N-1}^{(2)}$. These g_0-subalgebras are naturally associated in the T-dual picture to D-branes located at a specific orientifold fixed plane. Indeed, distinguishing the two cases $A_{2k-1}^{(2)}$ and $A_{2k}^{(2)}$ we find the g_0-subalgebras are C_k and B_k respectively [21, 22], indicating that the Dp-branes are located at the Op^+ plane and $\widetilde{O}p^-$ plane respectively, when no Wilson lines are turned on in the T-dual $o(p+1)$ picture.

We can now turn on Wilson lines. An interesting configuration arises when we turn on the Wilson lines which correspond in the T-dual picture to putting the D-branes at the position of the other orientifold p-plane. These Wilson lines are (denoting the compactified direction 4, and the gauge field corresponding to the massless vector field in this direction by A_4):

$$\begin{aligned} A_4 &= \check{\alpha}_1 + 2\check{\alpha}_2 + \cdots + (k-1)\check{\alpha_{k-1}} + k\check{\alpha}_k\,, \\ A_4 &= \check{\alpha_{k-1}} + 2\check{\alpha_{k-2}} + \cdots + (k-i)\check{\alpha}_i + \cdots + (k-1)\check{\alpha}_1 + \frac{k}{2}\check{\alpha}_0\,, \end{aligned}$$

respectively[8] [9]. These Wilson lines give rise to a mass for one of the open string states corresponding to the simple root α_k, respectively α_1, but not to the states corresponding to the other simple roots. Indeed, the states corresponding to the other simple roots are not charged under the $U(1)$ associated to the Wilson line, as can be seen by calculating their intersection with the Wilson line A_4 (for example $\langle \alpha_2, A_4 \rangle = 0$), but the simple root α_k (respectively α_0) is. Moreover, there are anti-periodic states which now become periodic (because of their charge under the $U(1)$ associated to the Wilson line). They give rise to a re-enlarging of the symmetry group that is appropriate for the orientifold plane at which the Dp-branes are positioned. Among these new extra massless states is the simple root corresponding to the root that is needed to complete the g_0 algebra to the full twisted affine algebra. These read in the two cases that we distinguish: θ_{hs} and $2\theta_{hs}$ where θ_{hs} is the highest

[7]All momenta are integer — we define them relative to the covering circle of length $2\pi R$. The interval has length πR.

[8]The numbering of the nodes in figure 4 for the g_0-subalgebras agrees with the standard conventions of [25], except for the $A_{2k}^{(2)}$ case. This is natural from the viewpoint of affine algebras since for all algebras but this one, the g_0 algebra is obtained by deleting the zeroth node of the Dynkin diagram (yielding the algebra known as g^o) [21].

[9]One way to find the appropriate Wilson lines is by looking at the positions of the D-branes and their mirrors in the covering theory, and then follow these Wilson lines through the construction of the twisted affine algebras [21, 22].

short root of the g_0-subalgebra. It is then straightforward to check (using the Cartan matrices of the g_0-subalgebras) that indeed the periodicity of the state associated to these roots changes from anti-periodic to periodic, by computing their charge under the $U(1)$ associated to the Wilson line. It is not difficult to infer the fate of the massless vector states corresponding to non-simple roots from the analysis for the massless vector states associated to the simple roots.[10] It is moreover easy to see how our analysis generalizes to other positions of the Dp-branes on the interval.

It would be interesting to have a better understanding of the geometric action of the outer automorphism. It is clear that it acts as an orientation reversal in spacetime, but it would be nice to have an interpretation of the associated signs (see e.g. [23]) in terms of the background fields in string theory.

4.2 $(\widetilde{O}^-, \widetilde{O}^-)$ orientifold planes

The twisted algebra $D^{(2)}_{k+1}$ arises differently. From the viewpoint of the parent theory, the Wilson line corresponding to positioning all the D-branes at one of the $\widetilde{O}^-$-planes, say the one at the origin, reads:

$$\mathrm{diag}(-\pi R, 0, 0, \ldots, 0, \ldots, 0)\,, \tag{4.3}$$

where in the parent theory we have $1+l$ D-branes corresponding to the $2\,\frac{1}{2}$ D-branes stuck to the orientifold planes (in positions 1 and $k+2$ in the diagonal matrix in (4.3)) and l branes and their mirrors. From the perspective of the parent theory, we can see that the massless states corresponding to this positioning of Dp-branes, and the positioning corresponding to the T-dual Wilson line

$$\mathrm{diag}(-\pi R, -\pi R, \ldots, -\pi R, 0, \pi R, \pi R, \ldots, \pi R) \tag{4.4}$$

gives rise to the expected symmetry enhancements. These can be reproduced by considering massless open string vector states as in the previous section, but corresponding to the $D^{(2)}_{k+1}$ algebra. By turning on Wilson lines purely within this algebra one cannot reproduce the same analysis as in the previous subsection,[11] but the embedding of the states in the parent theory remains clear, and we can analyze the charges of the states in the $D^{(2)}_{k+1}$ algebra under the Wilson line (4.4) that includes a U(1) piece. Then the analysis runs as before. In this case the g_0-subalgebra is B_k, and the algebra obtained after deleting the k-th node and adding in the zeroth node is B_k, once again as expected in a $(\widetilde{O}^-, \widetilde{O}^-)$ background. (Another way to analyze this case is by copying the first part of the analysis done for $A^{(2)}_{2k}$ and invoking the Z_2 symmetry of the background.)

[10] It is amusing that for the $A^{(2)}_{2k}$ case, there are now string modes with half-integer momentum, or T-dual winding, as expected from the fact that strings can end on the $\widetilde{Op}^-$-plane.

[11] The basic reason why the algebraic interpretation of the parent theory is a bit less straightforward in this case, is because of the non-trivial role of the center of mass U(1) which is not incorporated in a standard fashion in the algebraic framework.

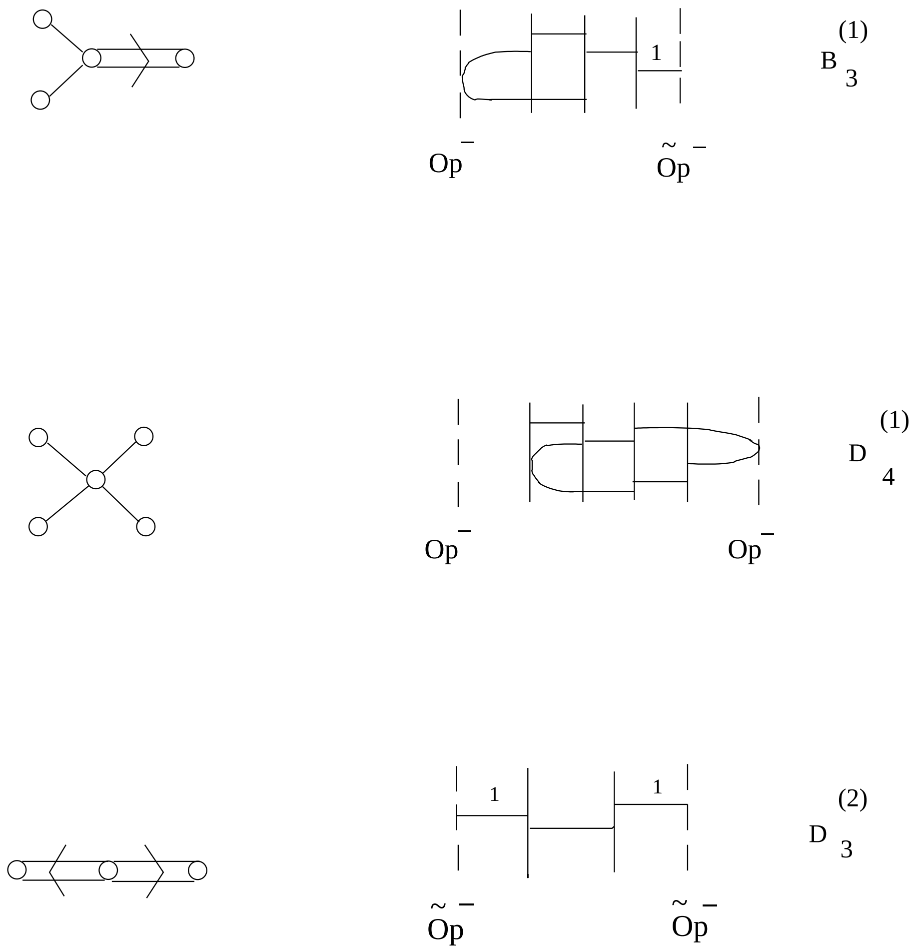

Figure 6: A few examples of combinations of orientifold planes with a minimum number (l of figure 4) of parallel D-branes between them.

4.3 Remarks

We showed the connection between the infinite algebras and the enhanced gauge symmetries that arise in the orientifold backgrounds. Before we delve into some details of more explicitly exhibiting the solutions in the low-energy theories corresponding to $D(p-2)$-branes spanning between Dp-branes, we add some general remarks.

First of all, since 16 supercharges are preserved, the vector state we discussed before should be thought of as being completed with fermionic and bosonic states to form a full supermultiplet, and all other states will similarly fall in supersymmetry multiplets. Next, when the interval has a finite width, not all of the vector states can become massless at the same time. The vector states we discussed will form an algebra which does not yield a symmetry group, but an infinite spectrum generating algebra, which should enable us to classify open string states at different mass levels into representations of the affine algebras. We have already one example in the

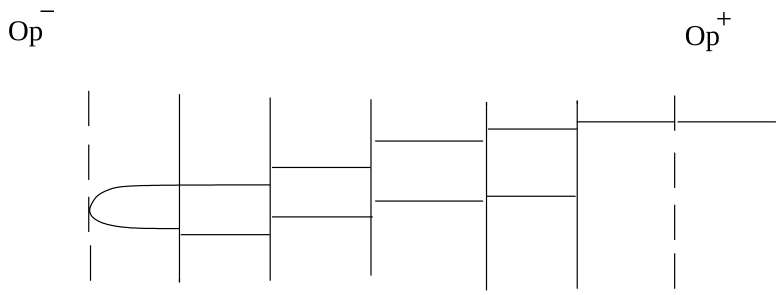

Figure 7: We draw each $F1$-string with a multiplicity given by the corresponding Kac label for the example $A_{2k-1}^{(2)}$ given in the text. We replaced the label '2' over the $F1$-string near the O^+ plane with an equivalent picture. The fundamental $F1$-strings with these multiplicities make up a string that wraps the covering circle.

form of the 1/2 BPS states in this background, associated to the $D(p-2)$ branes stretching between the Dp-branes. They are in one-to-one correspondence with the roots of the infinite algebra, where the Lie algebraic part indicates the brane they start from and end on, and where the imaginary part indicates their winding number (or T-dual momentum). The imaginary unit root, for example, corresponds to the $D1$-brane fully wrapping the circle,[12] as can be seen from the fact that it corresponds to the sum of all simple roots, weighted with the Kac marks of the affine algebra. (For example, in the $A_{2k-1}^{(2)}$ case, we have Kac marks [21] $(1,1,2,2,\ldots,2,1)$ and if we glue together fundamental $F1$-strings with these multiplicities, we find that they correspond to a $F1$-string going all the way around the circle (see figure 7). This gives a nice pictorial interpretation to the Kac marks of affine algebras.)

Using the vertex operators associated to the open string vector states, it should be possible to build a full affine multiplet on any perturbative open string state.

We also remark that there are more affine algebras than string theory backgrounds that we discussed. Some special low-rank cases correspond to a small number of D-branes between orientifold planes, and we can find the correspondence to affine algebras analogously to what we did before. We find for instance: $D_3^{(1)} = A_3^{(1)}$, $D_2^{(1)} = A_1^{(1)}$ (as in the work [34]), and $C_1^{(1)} = A_1^{(1)} = D_2^{(2)}$, and the $k=1$ case of $A_{2k}^{(2)}$ indeed corresponds to the algebra $A_2^{(2)}$. But the exceptional cases and the $D_4^{(3)}$ cases are left without a string theory picture. It is tempting to speculate that the last $D_4^{(3)}$ could arise from a Z_3 action on a covering string theory. Constructing backgrounds corresponding to these algebras is an open problem.

We now turn to constructing explicitly some of the $D(p-1)$ branes as generalized fundamental magnetic monopole solutions in the low-energy field theory on the $D(p+1)$-branes.

[12]Except for the $D_{k+1}^{(2)}$ case, where it spans the interval. Note that a string can start and end on the Dp-branes stuck to the orientifold $\widetilde{O}^-$ planes at the end of the intervals in this case.

5. Field theory

We reviewed in section 2 how $D1$-branes stretching between $D3$-branes can be viewed as monopole solutions in the $D3$-brane worldvolume gauge theory. After compactification and T-dualizing along the direction in which the $D3$-branes are separated (the 4 direction), these solutions become solutions to the self-duality equations for the field strength on $D4$-branes spanning $R^3 \times S^1$. Note also that after T-duality a $D1$-brane that originally wound the whole circle becomes a $D0$-brane embedded in $D4$-branes, which in the low-energy theory corresponds to an instanton in the gauge theory on the $D4$-brane [35]. We conclude [26], [36]–[40] that periodic instantons are made up of fundamental magnetic monopoles. The low-energy descriptions for the fundamental monopoles have been explicitly constructed in [40] for all non-twisted affine groups and we will generalize this construction to the twisted groups. The existence of these solutions was mentioned in the mathematical treatment [26].

In section 5.1 we review the well known magnetic monopole solutions corresponding to $su(2)$ subalgebras [41]–[44]. In section 5.2 we show how to find new explicit solutions to the self-duality equation on $\mathbb{R}^3 \times S^1$ with twisted boundary conditions, generalizing and simplifying [38, 40]. In section 5.3 we discuss properties of these new solutions that identify them as fundamental monopoles [18, 40, 45, 46], and mention how they can be used to construct (twisted) periodic instantons, building on the works [26, 36, 38, 39, 47, 48, 49].

5.1 Review

In this subsection we follow closely the line of development in [40]. We first exhibit the magnetic monopole solutions corresponding to the $su(2)$ embeddings. We parameterize the space $R^3 \times S^1$ by $x^{1,2,3}$ and $x^4 \equiv x^4 + 2\pi R$. The self-duality equation

$$*F = F \tag{5.1}$$

on $\mathbb{R}^3 \times S^1$ can be rewritten in terms of the fields[13] A_i $(i = 1, 2, 3)$ and $A_4 \equiv \Phi$ as a generalized monopole equation:[14]

$$B_i = D_i\Phi - \partial_4 A_i\,, \tag{5.2}$$

where $D_i = \partial_i - i[A_i, .]$. We choose the Wilson line (or T-dual scalar) vacuum expectation value (at $x^3 = -\infty$) to be:

$$\Phi_0 = v \tag{5.3}$$
$$= \phi_0 + \langle v, \alpha\rangle t^3\,, \tag{5.4}$$

[13] See also appendix A for our conventions.

[14] It is useful to interpret it as a monopole equation with structure group the loop group [26]. We will postpone making use of this interpretation to the next subsection.

where $v \in \mathcal{H}$, the t_i form a preferred $su(2)$ subalgebra of the gauge group (see appendix A) and we have

$$t^3 = \frac{1}{2}\check{\alpha} \tag{5.5}$$

$$\phi_0 = v - \frac{1}{2}\langle v, \alpha\rangle\check{\alpha}. \tag{5.6}$$

The value for ϕ_0 is chosen such that $[\phi_0, t^i] = 0$. The monopole solution corresponding to this $su(2)$ embedding is:

$$A_i = A(r,u)\epsilon_{ijk}e_r^j t^k \tag{5.7}$$

$$\Phi = \phi_0 + \phi(r,u)e_r^i t_i\,, \tag{5.8}$$

where we made use of the definitions

$$A(r,u) = \frac{1}{r} - \frac{u}{\sinh(ur)} \tag{5.9}$$

$$\phi(r,u) = \frac{1}{r} - u\coth(ur) \tag{5.10}$$

$$e_r^i = \frac{(x^1, x^2, x^3)}{r} \tag{5.11}$$

and the value for $u = \langle v, \alpha\rangle$ is chosen to make sure that the boundary conditions at infinity are satisfied. The topological charge of the monopole is given by:

$$m = \frac{1}{8\pi^2}\int_0^{2\pi R} dx^4 \int_{S^2_\infty} dS_i \left(B_i^a\Phi^a - \frac{1}{2}\epsilon_{ijk}A_j^a\partial_4 A_k^a\right), \tag{5.12}$$

where we recognize a contribution from the magnetic field at infinity, and a seemingly pure gauge part from the dependence of the gauge field on the circle coordinate. The solutions in this subsection did not involve the coordinate x^4 in a non-trivial way, and the field $A_4 = \Phi$ played the usual role of the adjoint Higgs scalar. In this context it was shown that the solutions along $su(2)$ embeddings which correspond to simple roots are fundamental[15] [18, 45, 46].

5.2 Extra solutions on a compact space

Now we turn to the extra solutions corresponding to the fact that we work on $\mathbb{R}^3 \times S^1$. In section 2 we already pointed out that we expect one more fundamental monopole solution to arise on this space. This is the solution we will concentrate on.

On the compact space, we have to specify our boundary conditions. The fields can be periodic:

$$A_\mu(x^4 + 2\pi R) = A_\mu(x^4)\,. \tag{5.13}$$

[15]The simple roots are chosen such that v lies in the fundamental chamber.

or they can be periodic up to an outer automorphism τ

$$A_\mu(x^4 + \frac{2\pi R}{k}) = A_\mu^a(x^4)\tau(T_a)\,, \tag{5.14}$$

where $\tau^k = 1$. The first case was treated in detail in [40]. There the extra fundamental monopole solution was found to be:

$$\begin{aligned}\Phi(x,\theta_0) &= \phi_0 + \frac{1}{2R}\check{\theta}_0 + e^{-\frac{ix^4}{R}t^3(-\theta_0)}t^i(-\theta_0)e^{\frac{ix^4}{R}t^3(-\theta_0)}e_r^i\phi(r,u_n),\\ A_i(x,\theta_0) &= e^{-\frac{ix^4}{R}t^3(-\theta_0)}t^l(-\theta_0)e^{\frac{ix^4}{R}t^3(\theta_0)}\epsilon_{ijl}e_r^jA(r,u_n)\,.\end{aligned} \tag{5.15}$$

Here we introduced the root θ_0. This root is related to the root $\alpha_\epsilon = \delta - \theta_0$, where δ is the imaginary unit root of the affine algebra, and α_ϵ is the root needed to complete the simple root system of the subalgebra g_0 to a simple root system of the full affine algebra. We moreover used the definition $u_n = \langle v, -\theta_0\rangle + \frac{1}{R}$. Now, we have chosen our normalization in (5.14) such that this solution also satisfies the boundary conditions for the twisted case, and one can check that this solution still satisfies the right boundary conditions for the Higgs field at infinity and the generalized monopole equation (5.2). To extend (and in the end simplify) the analysis of the properties of these solutions, it will be useful to introduce a few more facts about affine algebras first, then to rewrite the solutions in a more convenient and recognizable form.

Note that for the untwisted algebras θ_0 is θ_h, the highest root in g^o (see footnote 8), and for the twisted algebras θ_0 is the highest weight of the g_0 representation spanned by the roots in g_1 [21]. It will be convenient (as in the analysis of the affine Weyl group [22]) to introduce an extended version of the vacuum expectation value v. We define $v_e = v + d$, where d is the derivation of the affine algebra (normalized such that $\langle d, \delta\rangle = \frac{1}{R}$). We moreover define the $su(2)$ subalgebra of the affine algebra corresponding to the root α_ϵ as follows:

$$\begin{aligned}t_e^1(\alpha_\epsilon) &= \frac{1}{2}(E^1_{-\theta_0} + E^{-1}_{\theta_0}),\\ t_e^2(\alpha_\epsilon) &= \frac{1}{2i}(E^1_{-\theta_0} - E^{-1}_{\theta_0}),\\ t_e^3(\alpha_\epsilon) &= -\frac{1}{2}\check{\theta}_0\,,\end{aligned} \tag{5.16}$$

where $E^1_{-\theta_0} = e^{-i\frac{x^4}{kR}} \otimes E_{-\theta_0}$ (see e.g. [24]). These definitions allow us to rewrite the solution (5.15) in the standard form where $\phi_0^e = v - \langle v_e, \alpha_\epsilon\rangle t_3^e$ and $u_n = \langle v_e, \alpha_\epsilon\rangle$:

$$\begin{aligned}\Phi(x,\alpha_\epsilon) &= \phi_0^e + t_e^i(\alpha_\epsilon)e_r^i\phi(r,u_n),\\ A_i(x,\alpha_\epsilon) &= t_e^k(\alpha_\epsilon)\epsilon_{ijk}e_r^jA(r,u_n)\,.\end{aligned} \tag{5.17}$$

This simple observation will make life easier on us in the next section, and we already understand a lot better how to interpret the solutions found in [40] in a roundabout way[16]. Indeed, the solutions correspond to the $su(2)$ subalgebra associated to the simple root α_ϵ, embedded in the full affine algebra, much as the fundamental monopole solutions in subsection 5.1.

5.3 Fundamental monopoles

To prove that the monopoles constructed in the previous section are fundamental (in the sense of [18]), we have to analyze the number of zero modes of the equation

$$(-i\sigma_k D_k + D_4)\psi = 0 \tag{5.18}$$

in the background of the monopoles. The case with periodic boundary conditions has been treated in [40]. We will simplify and extend this analysis here. We find that the monopoles associated to the simple roots, and with a vacuum expectation value v have four zero modes (and thus are fundamental), when v lies in the fundamental alcove of the (twisted) affine algebra, i.e. when $\langle v, \alpha_i \rangle$ and $\frac{1}{R} - \langle v, \theta_0 \rangle$ are positive.[17] These conditions can be re-formulated as: $\langle v_e, \alpha_j \rangle$ is positive for *all* simple roots α_j of the affine algebra.

All this follows from a careful analysis of the zero mode equation, in analogy with the analysis in appendix C of the first reference in [18]. A crucial observation is that on the circle, we can expand the fluctuation mode in a basis for the affine algebra, where the KK-momentum around the circle correspond to the 'level' of the generators (as in the definition after equation (5.16)). We can then define a commuting isospin t^3 and hypercharge y_{tot} (where β should now be read as taking values in the root lattice of the affine algebra):

$$\begin{aligned} t_e^3 E_\beta &= [t_e^3, E_\beta] = \left[\frac{\check{\alpha}_\epsilon}{2}, E_\beta\right] \\ &= \frac{1}{2}\langle \check{\alpha}_\epsilon, \beta \rangle E_\beta \end{aligned} \tag{5.19}$$

and y_{tot}

$$\begin{aligned} y_{tot} &\equiv \frac{1}{\langle v_e, \alpha_\epsilon \rangle}[v_e - \langle v_e, \alpha_\epsilon \rangle t_e^3, E_\beta] \\ &= \frac{\langle v_e, \beta \rangle}{\langle v_e, \alpha_\epsilon \rangle} - \frac{1}{2}\langle \check{\alpha}_\epsilon, \beta \rangle. \end{aligned}$$

As in [18] we can then split the zero mode equation into multiplets with different hypercharge y_{tot} and isospin quantum numbers. The rest of the proof for the fact that the monopoles we discussed are fundamental then follows in complete parallel

[16] The method in [40] can be seen as being based on a symmetry between the zeroth root in the Dynkin diagram of an untwisted affine algebra and another, simple root in the Dynkin diagram.

[17] We treat only the case where the symmetry group is completely broken to $U(1)$ factors [18].

to the proof in appendix C of [18]. Crucial in the proof is only the fact that α_ϵ is simple. This implies that the hypercharge is larger than the isopsin ($|y| > t$) for all $\beta \neq \alpha_\epsilon$, and in that case it can be shown that there is no normalizable solution to the zero mode equation [18].

Note that a periodic instanton, with zero magnetic charge and instanton number 1 can be formed by appropriately [39, 49] gathering fundamental monopoles. For any affine algebra the multiplicity of the fundamental monopoles in a periodic instanton is given by the Kac marks (as pointed out for the untwisted algebras in [40]). The pictorial explanation for this we already gave: this corresponds to combining $D(p-2)$-branes until they fully wrap the circle (see figure 7). Note that, as before, we see a one to one correspondence between magnetic monopoles corresponding to $su(2)$ embeddings, and the co-roots of the (twisted) affine algebra. Simple co-roots correspond to fundamental monopoles, imaginary roots to periodic instantons, etcetera.

6. Conclusion and discussion

We pointed out a nice correspondence between orientifold string theory backgrounds and (twisted) affine algebras. Apart from clarifying string theory through algebra, it perhaps also clarifies twisted affine algebras to string theorists. Note that we had to introduce a new kind of orientifold action in compact spacetimes, including an outer automorphism action on Chan-Paton factors. This kind of projection might well generalize to more generic compact spacetimes. A first extension of our program in this paper might lie in the treatment of spacetimes including a $T^2/\mathbb{Z}_2$ factor with orientifold planes at the fixed points. The planar positioning of the orientifold planes seems to indicate a connection to more general Dynkin diagrams, of hyperbolic or generalized Kac-Moody algebras, although at present we do not know how to make this intuition precise.

Our correspondence moreover paves the way for an application in field theory. The twisted boundary conditions that we imposed, can be taken over in supersymmetric field theories on, for instance $\mathbb{R}^3 \times S^1$. One interesting recent result is the determination of the exact elliptic superpotential for the SU(N) $\mathcal{N} = 1^*$ theory on this space [50]. In that paper, a relation between general $\mathcal{N} = 1^*$ theories on $\mathbb{R}^3 \times S^1$ was pointed out. From the results on integrable models and $N = 2$ field theories summarized in [57], one concludes that the relevant integrable models [51, 52, 53] for general gauge groups with periodic boundary conditions are the twisted elliptic integrable models, since they yield the right superpotential [54, 55, 56] in the limit where one decouples the three chiral multiplets completely.[18] Our work indicates that the other elliptic superpotentials and their limiting affine Toda system [57]

[18]We thank Prem Kumar for many enlightening discussions in which this conclusion was drawn.

describe $\mathcal{N} = 1^*$ theories with twisted boundary conditions on the circle, and the corresponding $\mathcal{N} = 1$ theories. One should be able to check this from a first principle calculation of the superpotential along the lines of [56]. It should also be rewarding to delve more deeply into index theorems [46, 58, 59] on $\mathbb{R}^3 \times S^1$ with twisted boundary conditions to complete the zero mode analysis for twisted periodic instantons, where, for example, we would expect the number of zero modes of the instanton with charge one to be given by the Coxeter number of the affine algebra (instead of the dual Coxeter number for the usual periodic instantons).

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A. Conventions

The Yang-Mills field strength and magnetic field are defined as follows:

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - i[A_\mu, A_\nu]^a \\ B_i &= \frac{1}{2}\epsilon_{ijk}F^{jk} \,. \end{aligned} \tag{A.1}$$

For algebra conventions, we follow [21, 22], and define the set of simple roots as $\Pi = \{\alpha_1, \ldots \alpha_n\} \subset \mathcal{H}^*$ and the set of simple co-roots as $\Pi = \{\check{\alpha}_1, \ldots, \check{\alpha}_n\} \subset \mathcal{H}$. These are subsets of dual spaces, and we have

$$\langle \check{\alpha}_i, \alpha_i \rangle = a_{ij} \tag{A.2}$$

with a_{ij} the Cartan matrix of the algebra. A special basis is given by:

$$\begin{aligned} [H, E_{\alpha_i}] &= \langle H, \alpha_i \rangle E_{\alpha_i} \\ [E_\alpha, E_{-\alpha}] &= \check{\alpha} \,, \end{aligned} \tag{A.3}$$

where H denotes any element of the Cartan subalgebra (including the co-roots). We define $su(2)$ subalgebras associated to roots α by:

$$t^1(\alpha) = \frac{1}{2}(E_\alpha + E_{-\alpha})$$

$$t^2(\alpha) = \frac{1}{2i}(E_\alpha - E_{-\alpha})$$
$$t^3(\alpha) = \frac{1}{2}\check{\alpha}\,. \qquad \text{(A.4)}$$

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